A new method for calculating the Cartan forms and applications to the gauge and chiral field theories

1993 J. Phys. A: Math. Gen. 26 631

(http://iopscience.iop.org/0305-4470/26/3/022)

J. Phys. A: Math. Gen. 26 (1993) 631-645. Printed in the UK

A new method for calculating the Cartan forms and applications to the gauge and chiral field theories

V I Kuvshinov and Nguyen Vien Tho

Institute of Physics, Academy of Sciences of Belarus, 70 Skarina Avenue, Minsk 220602, Belarus

Received 6 May 1992

Abstract. We develop a method for calculating the Cartan forms in the parametrization of group transformations which has a simple composition law and satisfies the condition called the naturalness. The method is applied to gauge theories for finding the explicit transformation laws of gauge fields under finite local transformations of groups, and to chiral field theories for setting up chiral field Lagrangians. The explicit forms of finite transformations of gauge fields and their nonlinear realizations for gravity and supergravity are found. The expressions of the Cartan forms, the Lagrangians of the principal chiral fields and the Goldstone fields for the unitary groups U(2), SU(2), U(3), SU(3) and the coset spaces associated with these groups are derived. The Lagrangians obtained are distinguished from the previous known ones by new types of nonlinearity.

1. Introduction

It is well known that the effectivity of the group-theoretical methods essentially depends on the choice of parametrization. The vector parametrization, proposed at first for the rotation group O(3) and the Lorentz group [1, 2], and generalized afterwards for some other groups [3-5], has been convenient and effective in the investigation of many problems of these groups. The important features of this parametrization which are not present in many other parametrizations are the following: (i) The composition law of parameters which correspond to group multiplication has a simple form; (ii) The parametrization satisfies the condition, called naturalness; that is if $G(Q)$ is a group element, Q is the set of parameters, regarded as a vector in some space, then

$$G(Q=0)=I \qquad G(-Q)=G^{-1}(Q) \tag{1}$$

(I is the unit element of the group); (iii) The parametrization has also linearity; that is an inner automorphism corresponds to a linear transformation of parameters:

$$G(Q)G(Q')G^{-1}(Q)=G[\Lambda(Q)Q'] \tag{2}$$

(Λ is a matrix). Because of these properties many results can be obtained directly by using only operations on parameters and without addressing matrix forms of transformations of groups or their representations.

In the following we develop a method for calculating the Cartan forms in the vector parametrization of groups (section 2). It turns out that the calculation can be based only on the composition law of parameters and the naturalness of the parametrization, unlike in the usual approach based on solving the Cartan-Mauer's equations [6, 7].

0305-4470/93/030631+15$07.50

The method is applicable to many different groups and has many interesting applications. The applications are discussed in two directions: (i) finding the explicit transformation laws of gauge fields for gravity and supergravity under the finite local group transformations, the explicit form of the nonlinear realization of gauge fields (section 3); (ii) setting up the principal chiral and Goldstone field Lagrangians related to the unitary groups U(2), SU(2), U(3), SU(3) and the coset spaces associated with these groups (section 4). In the conclusion (section 5) we discuss the results obtained.

2. Method for calculating the Cartan forms

Let the Lie group G be parametrized by the set of parameters $Q=Q_n$. Suppose that (a) A composition law of parameters $Q''=\langle Q, Q'\rangle$ which is defined by the group multiplication $G(Q'')=G(Q)G(Q')$ is given, and Q'' continuously depends on Q, Q'; (b) the parametrization satisfies the naturalness (in the sense of formula (1)). From the viewpoint of geometry of group space of parameters an infinitely small vector $\mathrm{d}Q$ which has origin at the point Q corresponds to the group element $G(Q')$: $G(Q')G(Q)=G(Q+\mathrm{d}Q)$, or $G(Q')=G(Q+\mathrm{d}Q)G^{-1}(Q)$. Decomposing $G(Q+\mathrm{d}Q)$ to series in $\mathrm{d}Q$:

$$G(Q+\mathrm{d}Q)=G(Q)+\frac{\partial G}{\partial Q_m}\bigg|_Q \mathrm{d}Q_m+\frac{1}{2}\frac{\partial^2 G}{\partial Q_n \partial Q_m}\bigg|_Q \mathrm{d}Q_m\,\mathrm{d}Q_n+\ldots$$
$$=G(Q)+\mathrm{d}G(Q)+\mathrm{O}(\mathrm{d}Q^2)$$

we obtain

$$G(Q')=G(Q+\mathrm{d}Q)G^{-1}(Q)=I+(\mathrm{d}G(Q))G^{-1}(Q)+\mathrm{O}(\mathrm{d}Q^2). \tag{3}$$

In the other side, from the naturalness (1) we can represent $G(Q')$ in the form

$$G(Q')=G(Q+\mathrm{d}Q)G^{-1}(Q)=G(Q+\mathrm{d}Q)G(-Q) \tag{4}$$

and therefore

$$Q'=\langle Q+\mathrm{d}Q,-Q\rangle. \tag{5}$$

In equation (5) Q' *continuously depends on* $\mathrm{d}Q$ (*by the preposition* (a)) *and when* $\mathrm{d}Q=0$ we have $Q'=\langle Q,-Q\rangle=0$ (by the naturalness, preposition (b)), therefore the parameters Q'_n are infinitely small and can be represented in the form

$$Q'_n=\langle Q+\mathrm{d}Q,-Q\rangle_n=\alpha_{nm}(Q)\,\mathrm{d}Q_m+\mathrm{O}(\mathrm{d}Q^2)=Q'^{(1)}_n+\mathrm{O}(\mathrm{d}Q^2) \tag{6}$$

where the coefficients $\alpha_{nm}(Q)$ depend on the concrete form of the composition formula, $Q'^{(1)}_n=\alpha_{nm}(Q)\,\mathrm{d}Q_m$ is the first order (in $\mathrm{d}Q$) terms in the expression of Q'_n (6).

Remember that the tangent space to the Lie group G in the unit element of the group is identified with the Lie algebra of this group, $G(Q')$ have the form

$$G(Q')=I+\mathrm{i}Q'^{(1)}_n X_n+\mathrm{O}(\mathrm{d}Q^2) \tag{7}$$

where X_n are the generators of the group G. Comparing (3) and (7) we obtain

$$(\mathrm{d}G(Q))G^{-1}(Q)=\mathrm{i}Q'^{(1)}_n X_n. \tag{8}$$

From (8) it is clear that the Cartan forms in this approach is just the first-order (in $\mathrm{d}Q$) terms ($Q'^{(1)}_n$) in the expression of the composition $\langle Q+\mathrm{d}Q,-Q\rangle$.

3. The finite transformation laws of gauge fields and their nonlinear realizations for gravity and supergravity

The explicit transformation laws of gauge fields under the local group transformations are written usually in infinitesimal form. It is not always easy to find explicit finite forms of these transformation laws. In [3, 8] such transformations have been found for the gauge group SU(2) and the Lorentz group. In these works the calculations were based on the matrix expression of group elements or representations. However, for the more complicated cases such as Poincaré groups and supergroups, it is practically not possible to go on this way. The method developed in section 2 helps us resolve the problem. In the following we show that by this method it is possible to obtain not only the explicit finite transformation laws of gauge fields for the Poincaré group and supergroup, but the explicit form of their nonlinear realizations as well.

3.1. The finite transformation laws of gauge fields of the Poincaré group

Let us consider the gauge theory of gravity in which the gauge group is the Poincaré group [9]. It is known that in terms of the geometry of a principal fibre bundle the gauge fields are connection coefficients on a cross-section of a bundle with spacetime as its base and the symmetry group as its fibre. A finite local transformation S of the gauge group (that is a finite group transformation whose parameters depend on the points of base spacetime) are described as a change of section. Under this transformation the connection Ω (the Lie algebra valued one-form on spacetime) is transformed as the following [10]:

$$\Omega' = S\Omega S^{-1} + S\,\mathrm{d}S^{-1}. \tag{9}$$

In the following we show that it is possible to find the explicit form of transformation laws of the coefficients Ω_μ^A of connection one-form Ω ($\Omega = \Omega_\mu^A I_A\,\mathrm{d}x^\mu$, I_A ($A = 1, \ldots, 10$) are generators of the Poincaré group, x_μ are coordinates of base spacetime) under the gauge transformation (9). We use the parametrization proposed in [4]. According to [4] the transformations of the Poincaré group can be represented in the 5×5-matrix form:

$$S = \begin{pmatrix} L^2(\boldsymbol{p}) & L(\boldsymbol{p})b/|1+\boldsymbol{p}^2| \\ 0 & 1 \end{pmatrix} \tag{10}$$

where $\boldsymbol{p} = \{p_a, p_a^*\}$ ($a = 1, 2, 3$) is a complex three-dimensional vector parameter; $b = \{b_m\}$ ($m = 1, 2, 3, 4$) is a real four-dimensional vector parameter, $L(\boldsymbol{p})$ is the 4×4 matrix of an arbitrary Lorentz transformation [2]:

$$L(\boldsymbol{p}) = \frac{(1+p_+)(1+p_-^*)}{|1+\boldsymbol{p}^2|} \qquad p_\pm = \begin{pmatrix} \boldsymbol{p}^\times & \pm\boldsymbol{p} \\ \mp\boldsymbol{p} & 0 \end{pmatrix} \tag{11}$$

$\boldsymbol{p}^\times$ is the antisymmetrical 3×3 matrix, dual with the vector $\boldsymbol{p}$, $(\boldsymbol{p}^\times)_{ab} = \varepsilon_{abc}p_c$. The composition law of parameters has the following form:

$$Q'' = \langle Q, Q'\rangle \qquad Q = \{\boldsymbol{p}, \boldsymbol{p}^*, b\} \qquad Q' = \{\boldsymbol{p}', \boldsymbol{p}'^*, b'\} \qquad Q'' = \{\boldsymbol{p}'', \boldsymbol{p}''^*, b''\}$$

$$\boldsymbol{p}'' = \frac{(1-\boldsymbol{p}'^2)\boldsymbol{p} + (1-\boldsymbol{p}^2)\boldsymbol{p}' + 2[\boldsymbol{p}\boldsymbol{p}']}{1+\boldsymbol{p}^2\boldsymbol{p}'^2 - 2\boldsymbol{p}\boldsymbol{p}'} \qquad \boldsymbol{p}''^* = (\boldsymbol{p}'')^*$$
$$b'' = \frac{|1+\boldsymbol{p}'^2|L(-\boldsymbol{p}'')L(\boldsymbol{p})b + |1+\boldsymbol{p}^2|L(\boldsymbol{p}'')L(-\boldsymbol{p}')}{|1+\boldsymbol{p}^2\boldsymbol{p}'^2 - 2\boldsymbol{p}\boldsymbol{p}'|}. \tag{12}$$

It is easy to show that the parametrization (10) is natural and linear in the sense of the formulae (1) and (2). It should be noticed that the usual parametrizations of the Poincaré group do not have these properties. The matrix $\Lambda(Q)$ in the formula (2) is the following

$$\Lambda(Q)=\begin{pmatrix} O^2(p) & 0 & 0 \\ 0 & O^2(p^*) & 0 \\ \alpha_+(p,p^*,b) & \alpha_-(p,p^*,b) & L^2(p) \end{pmatrix} \tag{13}$$

where $O(p)$ is the complex 3×3 matrix which acts in the space of complex parameters p and takes the form

$$O(p)=1+2\frac{p^\times+(p^\times)^2}{1+p^2} \qquad O\tilde{O}=1 \tag{13a}$$

$\alpha_\pm(p,p^*,b)$ are the rectangular 4×3 matrices whose elements are

$$[\alpha_\pm(p,p^*,b]_{ma}=-\frac{2}{|1+p^2|}[L^2(p)I_a^{(\pm)}L(-p)b]_m \tag{13b}$$

$I_a^{(\pm)}$ are the 4×4 matrices of generators of Lorentz group whose elements are

$$\begin{aligned}&[I_a^{(\pm)}]_{mn}=\tfrac{1}{2}[\pm(\delta_{ma}\delta_{4n}-\delta_{an}\delta_{m4})-\tfrac{1}{2}\varepsilon_{abc}(\delta_{cn}-\delta_{bn}\delta_{mc})]\\ &(a,b,c=1,2,3; \qquad m,n=1,2,3,4).\end{aligned} \tag{14}$$

In the five-dimensional representation (10) of the elements of the Poincaré group, ten group generators $I_A=\{J_a^{(\pm)},T_m\}$ are the following 5×5 matrices:

$$J_a^\pm=\begin{pmatrix} 2I_a^{(\pm)} & 0 \\ 0 & 0 \end{pmatrix} \qquad (T_m)_{MN}=\sigma_{mM}\delta_{N5} \qquad (M,N=1,2,\ldots,5). \tag{15}$$

The connection one-form Ω can be written as

$$\Omega=\Omega_A I_A=\Omega_{\mu A}I_A\,\mathrm{d}x^\mu=(\omega_{\mu a}J_a^+ +\omega_{\mu a}^* J_a^{(-)}+\theta_\mu^m T_m)\,\mathrm{d}x^\mu \tag{16}$$

and under the transformation (9) Ω_A transform as

$$\Omega_A' I_A=\Omega_B S(Q)I_B S(-Q)+S(Q)\,\mathrm{d}S^{-1}(Q). \tag{17}$$

Based on the linearity (2) it is easy to find that

$$S(Q)I_B S(-Q)=\Lambda_{AB}(Q)I_A$$

which defines the first term of (17) ($\Lambda(Q)$ is the matrix (13)). The second term is the Cartan forms which can be calculated from the composition law (12) by the method presented in section 2. After some manipulations we obtain for the transformations of the connection coefficients (gauge fields) under the group transformations with arbitrary finite local parameters $Q(x)=\{p(x),p^*(x),b(x)\}$:

$$\begin{aligned}\omega_{\mu a}' &= [O^2(p)]_{ab}\omega_{\mu b}+F_{ab}(p)\partial_\mu p_b\\ \omega_{\mu a}'^* &= [O^2(p^*)]_{ab}\omega_{\mu b}^*+F_{ab}^*(p)\partial_\mu p_b^*\\ \theta_\mu'^m &= [\alpha_+(p,p^*,b)]_a^m\omega_{\mu a}+[\alpha_-(p,p^*,b)]_a^m\omega_{\mu a}^*+[L^2(p)]^{mn}\theta_\mu^n\\ &\quad +[\beta+(p,p^*,b)]_a^m\partial_m u p_a+[\beta_-(p,p^*,b)]_a^m\partial_\mu p_a^*-\frac{1}{|1+p^2|}[L(p)]^{mn}\partial_\mu b^n\end{aligned} \tag{18}$$

where the matrices $O(p)$, $\alpha_\pm(p, p^*, b)$ are given by (13a), (13b)

$$F_{ab}(p) = -\frac{(1-p^2)\delta_{ab}+2p_a p_b - 2\varepsilon_{abc}p_c}{(1+p^2)}. \tag{18a}$$

The matrix $\beta_\pm(p, p^*, b)$ are the rectangular 4×3 matrices whose elements are

$$[\beta_+(p, p^*, b)]_{ma} = \frac{1}{|1+p^2|}\left\{\left[\frac{p_a}{1+p^2} - (4F_{ba}(p)+f_{ba}(p))I_b^{(+)}\right] L(p)b\right\}$$

$$[\beta_-(p, p^*, b)]_{ma} = \frac{1}{|1+p^2|}\left\{\left[\frac{p_a^*}{1+p^{*2}} - (4F_{ba}^*(p)+f_{ba}^*(p)I_b^{(-)})\right] L(p)b\right\}_m \tag{18b}$$

$$f_{ab}(p) = 2\frac{\delta_{ab}+\varepsilon_{abc}p_c}{1+p^2}.$$

From the transformation formulae (18) it is difficult to determine the tensor nature (with respect to the local frame) of the gauge field components. However, these field components can be replaced by their appropriate functionals which have clear tensor nature and are considered as independent field variables. Usually this is implemented by considering nonlinear realizations and introducing nonlinear gauge fields [9, 11, 12].

3.2. The nonlinear realization for the Poincaré group

Let us consider in the parametrization nonlinear realization for the Poincaré group which is linear in the Lorentz group. One can represent the transformations of the Poincaré group (10) in the form:

$$S = \begin{pmatrix} 1 & L(p)b/|1+p^2| \\ 0 & 1 \end{pmatrix}\begin{pmatrix} L^2(p) & 0 \\ 0 & 1 \end{pmatrix} = S_F S_L \tag{19}$$

where we denote with L the Lorentz subgroup and with F the coset space P/L. Then one can introduce the field $\Phi(x)$ which takes values in the coset space at a fixed point x:

$$\Phi(x) = \begin{pmatrix} 1 & L(p)\xi(x)/|1+p^2| \\ 0 & 1 \end{pmatrix} \in F_x \qquad \xi(x) \in R^{1,3} \tag{20}$$

and define the nonlinear gauge field $\hat{\Omega}$:

$$\hat{\Omega} = \Phi^{-1}\Omega\Phi + \Phi^{-1}\,\mathrm{d}\Phi. \tag{21}$$

Taking into account the explicit form (15) of generators of the Poincaré group, for the 5×5 matrices $\Omega, \hat{\Omega}$ can write

$$\Omega = \begin{pmatrix} 2\omega & \theta \\ 0 & 0 \end{pmatrix} \qquad \hat{\Omega} = \begin{pmatrix} 2\hat{\omega} & \hat{\theta} \\ 0 & 0 \end{pmatrix} \tag{22}$$

where $\omega, \hat{\omega}$ are one-forms which take the values in the Lie algebra of the Lorentz subgroup: $\omega = (\omega_{\mu a}I_a^{(+)} + \omega_{\mu a}^* I_a^-)\,\mathrm{d}x^\mu$, $\hat{\omega} = (\hat{\omega}_{\mu a}I_a^{(+)} + \hat{\omega}_{\mu a}^* I_a^{(-)})\,\mathrm{d}x^\mu$; θ, $\hat{\theta}$ are $R^{1,3}$-valued one-forms: $\theta = \theta_\mu^m I_m\,\mathrm{d}x^\mu$, $\hat{\theta} = \hat{\theta}_\mu^m I_m\,\mathrm{d}x^\mu$. The formulae (20) and (21) allow us to define $\hat{\theta}$, $\hat{\theta}$ from ω, θ (by using the naturalness, linearity and the method proposed in section 2)

$$\hat{\omega} = \omega \qquad \hat{\theta} = \theta + D\left(\frac{L(p)\xi}{|1+p^2|}\right) \qquad D = d + \omega. \tag{23}$$

Under the change of the cross-section $\hat{\omega}$ is transformed as the linear gauge field of the Lorentz group:

$$\hat{\omega}' = L^2(p)\omega(L^2(p)^{-1} + L^2(p)\,\mathrm{d}(L^2(p))^{-1} \tag{24}$$

and $\hat{\theta}$ as a Lorentz tensor: $\hat{\theta}' = L^2(p)\hat{\theta}$. Therefore $\hat{\omega}$ and $\hat{\theta}$ can be interpreted, respectively, as the local Lorentz connection and tetrads ($e^m_\mu = k_0\hat{\theta}^m_\mu$, $[k_0] = cm$). From the nonlinear gauge field $\hat{\omega}$ the culvature 2-form, the action and equations of motion of gravity may be constructed by the known procedures [9, 12].

3.3. *The finite transformation laws of gauge fields of the Poincaré supergroup*

Now let us go over to the case of supergravity formulated as a gauge theory of the Poincaré supergroup [12–14]. The natural and linear parametrization of the Poincaré supergroup has been proposed in [5]. In [5] the elements of supergroup $S(Q)$ are parametrized by the set Q of parameters: $Q = (p, p^*, B, A, A^*)$, where p is a complex three-dimensional vector parameter, B is a Hermitic 2×2 matrix, A is a two-dimensional complex vector with anticommuting components. The compositional law of parameters is the following:

$$Q'' = \langle Q, Q'\rangle$$

$$p'' = \langle p, p'\rangle = \frac{(1-p'^2)p + (1-p^2)p' + 2[pp']}{1+p^2p'^2 - 2pp'} \qquad p^{*\prime\prime} = (p'')^*$$

$$\begin{aligned} B'' = (1+p^2p'^2-2pp')^{-1}\{&|1+p'^2|l''^{-1}lBl^+l''^{+-1} + |1+p^2|l''l'^{-1}B'l'^{-1+}l''^+ \\ &- \mathrm{i}\sqrt{1+p^{2*}}\sqrt{1+p'^2}\,l''^{-1}\bar{l}A\cdot\bar{A}'l'^{-1+}l''^+ \\ &+ \mathrm{i}\sqrt{1+p'^{*2}}\sqrt{1+p^2}\,l''l'^{-1}A'\cdot\bar{A}l^+l''^{-1+}\} \end{aligned} \tag{25}$$

$$A'' = \sqrt{1+p''^2}\left(\frac{l''^{-1}lA}{\sqrt{1+p^2}} + \frac{l''l'^{-1}A'}{\sqrt{1+p'^2}}\right) \qquad A^{*\prime\prime} = (A'')^*$$

where

$$l \in SL(2,C) \qquad l = l(p) = \frac{(1+\hat{p})}{\sqrt{1+p^2}} \qquad \hat{p} = \mathrm{i}p_a\sigma_a,$$

σ_a are Pauli matrices

$$A\cdot\bar{A}' = \begin{pmatrix} A_1 \\ A_2 \end{pmatrix}\cdot(A_1^{*\prime}, A_2^{*\prime}) = \begin{pmatrix} A_1A_1^{*\prime} & A_1A_2^{*\prime} \\ A_2A_2^{*\prime} & A_2A_2^{*\prime} \end{pmatrix}.$$

The linear transformation which corresponds to the inner automorphism $S(Q'') = S(Q)S(Q')S^{-1}(Q)$ is

$$p'' = O^2(p)p' \qquad O(p) = 1 + 2\frac{p^\times + (p^\times)^2}{1+p^2} \qquad (p^\times)_{ab} = \varepsilon_{abc}p_c \qquad O\tilde{O} = I$$

$$\begin{aligned} B'' = l^2B'l^{+2} - |1+p^2|^{-1}[&2l^2\hat{p}'^2l^{-1}(B+\mathrm{i}A\cdot\bar{A})l^+ + 2l(B-\mathrm{i}A\cdot\bar{A})l^{-1+}\hat{p}'l^{+2}] \\ &- 2\mathrm{i}(1+p^2)^{-1/2}lA\cdot\bar{A}'l^{+2} - 2\mathrm{i}(1+p^{*2})^{-1/2}l^2A'\cdot\bar{A}l^+ \end{aligned} \tag{26}$$

$$A'' = l^2A' - 2(1+p^2)^{-1/2}l^2\hat{p}'l^{-1}A.$$

The connection one-form which takes values in the superalgebra has the following decomposition:

$$\Omega = \omega_a J_a^+ + \omega_a^* J_a^- + \theta_m T^m + \Psi_\alpha Q_\alpha^\times - \Psi_\alpha^* Q_\alpha^{*\times}$$
$$= (\omega_{\mu\alpha} J_\alpha^+ + \omega_{mu a}^* J_\alpha^- + \theta_{\mu m} T^m + \Psi_{\mu\alpha} Q_\alpha^\times - \Psi_{\mu\alpha}^* Q_\alpha^{*\times})\, dx^\mu$$
$$Q_\alpha^\times = \varepsilon_{\alpha\beta} Q_\beta. \tag{27}$$

Using the composition law (25), the linear transformation (26) and using similar methods as for the Poincaré group, we obtain for the transformations of the gauge fields under the supergroup transformations with arbitrary finite local parameters $Q(x) = \{p(x), p^*(x), B(x), A(x), A^*(x)\}$:

$$\begin{aligned} \omega'_{\mu\alpha} &= O^2_{ab}(p)\omega_{\mu b} + F_{ab}(p)\partial_\mu p_b \qquad \omega^{*\prime}_{\mu\alpha} = (\omega'_{\mu a})^* \\ \theta'_\mu &= l^2\theta_\mu l^{+2} - |1+p^2|^{-1}[2l^2\hat{\omega}_\mu l^{-1}(B + iA\cdot\bar{A})l^+ + 2l(B - iA\cdot\bar{A})l^{-1+}\hat{\omega}^+_\mu l^{+2}] \\ &\quad - 2i(1+p)^{-1/2} lA\cdot\bar{\Psi}_\mu l^{+2} - 2i(1+p^2)^{-1/2} l^2\Psi_\mu\cdot\bar{A}l^+ \\ &\quad + |1+p^2|(1+p^2)^{-2}[(\alpha_a+\beta_a)l(B+iA\cdot\bar{A})l^+\partial_\mu p_a \\ &\quad + l(B - iA\cdot\bar{A})l^+(\alpha_a^+ + \beta_a^+)\partial_\mu p_a^* \\ &\quad + l\partial_\mu B l^+ + ilA\cdot(\partial_\mu\bar{A})l^{-1} - il(\partial_\mu A)\cdot\bar{A}l^+] \\ \Psi'_\mu &= l^2\Psi_\mu - (1+p^2)^{-1/2}[2l^2\hat{\omega}_\mu l^{-1}A - (\alpha_a+\beta_a)A\partial_\mu p_a + l\partial_\mu A] \qquad \Psi^{*\prime}_\mu = (\Psi'_\mu)^* \end{aligned} \tag{28}$$

where

$$\theta_\mu = \theta_{\mu m}\sigma^m \qquad \theta'_\mu = \theta'_{\mu m}\sigma^m \qquad \hat{\omega}_\mu = i\omega_{\mu a}\sigma_a \qquad \hat{\omega}^+_\mu = -i\omega^*_{\mu a}\sigma_a$$
$$F_{ab}(p) = -(1+p^2)^{-2}[(1+p^2)\delta_{ab} + 2p_a p_b + (p^\times)_{ab}]$$
$$\alpha_a = (1+p^2)^{-1}p_a - i\sigma_a \qquad \beta_a = (1+p^2)^{-1}p_a - 2iF_{ba}\sigma_b$$

3.4. *The nonlinear realization for the Poincaré supergroup*

The nonlinear realization for the Poincaré supergroup which is linear on the Lorentz group is performed by the same scheme as for the Poincaré group.

A supergroup transformation can be represented in the form

$$\begin{aligned} S(p, p^*, B, A, A^*) &= S\left(0, 0, \frac{l(p)Bl^+(p)}{|1+p^2|}, \frac{l(p)}{\sqrt{1+p^2}}, \frac{l^*(p)A^*}{\sqrt{1+p^{*2}}}\right) S(p, p^*, 0, 0, 0) \\ &= S_F S_L \end{aligned} \tag{29}$$

where S_L is an element of the Lorentz subgroup, S_F is an element of the coset space G/L (we denote with L the Lorentz subgroup). One can define the nonlinear gauge field $\hat{\Omega}$

$$\hat{\Omega} = \Phi^{-1}\Omega\Phi + \Phi^{-1}\,d\Phi \tag{30}$$

where the field $\Phi(x)$ takes the values in the coset space in a fixed point x and has the form:

$$\begin{aligned} \Phi(x) &= S\left(0, 0, \frac{l(p)Bl^+(p)}{|1+p^2|}, \frac{l(p)\eta(x)}{\sqrt{1+p^2}}, \frac{l(p^*)\eta^*(x)}{\sqrt{1+p^{*2}}}\right) \\ \xi(x) &= \xi^m(x) \qquad \xi^m(x) \in R^{1,3} \qquad \eta(x) = \begin{pmatrix} \eta_1(x) \\ \eta_2(x) \end{pmatrix} \end{aligned} \tag{31}$$

$\eta_1(x)$, $\eta_2(x)$ are anticommuting components. From (30), (31) and the properties of the parametrization one can obtain for the explicit form of the nonlinear gauge fields:

$$\hat{\omega}_{\mu a}=\omega_{\mu a} \qquad \hat{\omega}^*_{\mu a}=\omega^*_{\mu a}$$

$$\begin{aligned}\hat{\theta}_\mu=\theta_\mu+\mathscr{D}_\mu\left(\frac{l(p)\xi(x)l^+(p)}{|1+p^2|}\right)-\mathrm{i}\left[\Delta^{\rightarrow}_\mu\left(\frac{l(p)\eta(x)}{\sqrt{1+p^2}}\right)\right]\cdot\bar{\eta}(x)\\ +\mathrm{i}\eta(x)\cdot\left[\left(\overline{\frac{l(p)\eta(x)}{\sqrt{1+p^2}}}\right)\Delta^{\leftarrow +}_\mu\right]\\ +2\mathrm{i}\left(\frac{l(p)\eta(x)}{\sqrt{1+p^2}}\right)\cdot\bar{\Psi}_\mu+2\mathrm{i}\Psi_\mu\cdot\left(\overline{\frac{l(p)\eta(x)}{\sqrt{1+p^2}}}\right)\end{aligned} \tag{32}$$

$$\hat{\Psi}_\mu=\Psi_\mu+\Delta^{\rightarrow}_\mu\left(\frac{l(p)\eta(x)}{\sqrt{1+p^2}}\right) \qquad \hat{\bar{\Psi}}_\mu=\bar{\Psi}_\mu+\left(\overline{\frac{l(p)\eta(x)}{\sqrt{1+p^2}}}\right)\Delta^{\leftarrow +}_\mu$$

where

$$\mathscr{D}_\mu=\partial_\mu+\hat{\omega}^{\rightarrow}_\mu+\hat{\omega}^{\leftarrow +}_\mu$$

$$\Delta^{\rightarrow}_\mu=\partial^{\rightarrow}_\mu+2\hat{\omega}^{\rightarrow}_\mu \qquad \Delta^{\leftarrow +}_\mu=\partial^{\leftarrow}_\mu+2\hat{\omega}^{\leftarrow +}$$

$$\hat{\omega}_\mu=\mathrm{i}\omega_{\mu a}\sigma_a \qquad \hat{\omega}^+_\mu=-\mathrm{i}\omega^*_{\mu a}\sigma_a \qquad \theta_\mu=\theta_{\mu m}\sigma^m \qquad \hat{\theta}_\mu=\hat{\theta}_{\mu m}\sigma^m.$$

Under the change of section the connection one-form $\hat{\Omega}$ is transformed as

$$\begin{aligned}&\hat{\omega}'_{\mu a}=O^2_{ab}(p)\hat{\omega}_{\mu b}+F_{ab}\partial_\mu p_b \qquad \hat{\omega}^{*\prime}_{\mu a}=(\hat{\omega}'_{\mu a})^*\\ &\hat{\theta}'_\mu=l^2(p)\hat{\theta}_\mu l^{+2}(p)\\ &\hat{\Psi}'_\mu=l^2(p)\hat{\Psi}_\mu \qquad \hat{\bar{\Psi}}'_\mu=\overline{(\hat{\Psi}'_\mu)}.\end{aligned} \tag{33}$$

The components $\hat{\omega}_{\mu a}$, $\hat{\omega}^*_{\mu a}$, $\hat{\theta}_{\mu m}$, $\hat{\Psi}_{\mu a}$ of the nonlinear gauge field $\hat{\Omega}$ with the transformation law (33) are interpreted as, respectively, local Lorentz connection, tetrads and Rarita–Schwinger field. In terms of these fields one can construct the culvatures, the action and the equations of motion of supergravity [12–14].

4. Lagrangians of the principal chiral and Goldstone fields associated with the unitary groups

Nonlinear chiral field models have been the subject of intense study in theoretical physics [10, 15, 16]. In these models the interaction is defined not by adding an interaction term to the free field Lagrangian, but by the geometrical structure of the nonlinear manifold in which the field takes values. For such chiral fields as the principal chiral fields, the n-fields, the Goldstone fields, the Skyrme fields, etc., the manifold M of field values is either a Lie group G or a coset space G/H (H is the subgroup of G). It is possible to set up the Lagrangians of the nonlinear chiral fields by considering the geometry of the group space which is analogous to the geometry of the Riemannian spaces and the spaces of affine connections. The fields are identified with the local parameters of group spaces or coset spaces, the Lagrangians are expressed through the Cartan forms for the corresponding Lie group space [6, 7].

4.1. *The vector parametrization of the unitary groups* U(2), SU(2), U(3), SU(3)

For deriving the Cartan forms associated with unitary groups by the method presented in section 2, one cannot use the exponential parametrization $U=\exp(\mathrm{i}\xi_k X_k)$ (X_k are generators), because the composition law of arbitrary finite parameters has not the simple form. An adequate one is the parametrization by the Caley form [17, 18]

$$U=U(N)=\frac{1+N}{1-N} \qquad U\in U(n) \tag{34}$$

where N is a antihermitic $n\times n$-matrix $N^{+}=-N$. The composition law of parameter matrices N is:

$$\begin{aligned} U(L)&=U(M)\,U(N),\, L=\langle M, N\rangle \\ &=1-(1-N)(1+MN)^{-1}(1-M). \end{aligned} \tag{35}$$

In particular, one has [18]:

$$\langle M+\delta N, M\rangle \simeq (1-M)^{-1}\delta N(1+M)^{-1} \tag{36}$$

where δN is a small addition to the matrix M. It is easy to verify that the parametrization has naturalness (1) and linearity (2).

Let us consider the unitary groups U(2) and U(3). Then instead of the matrix N one can choose as parameters of these groups the components of matrices M in σ-basis (for U(2)) or λ-basis (for U(3)), that is for U(2):

$$U=U(\eta)=\frac{1-\mathrm{i}\eta_i\sigma_i}{1+\mathrm{i}\eta_i\sigma_i} \tag{37}$$

where $\eta_i=(\eta_0, \eta)$ are four real parameters which make up a vector in R^4; $\sigma_0=I$, $\sigma_{1,2,3}$ are Pauli matrices, and for U(3):

$$U=U(\eta)=\frac{1-\mathrm{i}\eta_i\lambda_i}{1+\eta_i\lambda_i} \tag{38}$$

where $\eta_i=(\eta_0, \eta_1, \ldots, \eta_8)$ are nine parameters which make up a vector in R^9; $\lambda_0=\sqrt{\frac{2}{3}}I$, $\lambda_1, \ldots, \lambda_8$ are Gell-Mann matrices. For SU(2), the condition of unimodularity reduces to the fact that the parameter η_0 in (37) is equal to zero, therefore $U\in \mathrm{SU}(2)$ is parametrized by three real parameters which make up a vector in R^3, $\eta=(\eta_1, \eta_2, \eta_3)$:

$$U=\frac{1-\mathrm{i}\eta\sigma}{1+\mathrm{i}\eta\sigma}. \tag{39}$$

For SU(3), the condition of unimodularity leads to the nonlinear equation which connects nine parameters in (38):

$$3\eta_0^3-\eta_0\eta_i\eta_i+\mathrm{i}\sqrt{\tfrac{2}{3}}h_{jkl}\eta_j\eta_k\eta_l-\eta_0=0 \tag{40}$$

where h_{jkl} are constants which are completely defined by the structure constants of the group U(3) (see the formula (A.3) in the appendix).

4.2. *The expressions of the Cartan forms for the groups* U(2), SU(2), U(3), SU(3)

According to the method presented in section 2 and using the formula (36) one can obtain the Cartan forms for the group U(2):

$$\begin{aligned} (\mathrm{d}U)U^{-1}&=-2\mathrm{i}f_{ij}(\eta_0, \eta_1, \eta_2, \eta_s)\,\mathrm{d}\eta_j\sigma_i \\ f_{ij}(\eta_0, \eta_1, \eta_2, \eta_3)&=\frac{\delta_{ij}+\alpha^{(1)}_{ijk}\eta_k+\alpha^{(2)}_{ijkl}\eta_k\eta_l}{(1-\eta_0^2+\eta^2)^2+4\eta_0^2} \end{aligned} \tag{41}$$

where $\alpha^{(1)}_{ijk}$, $\alpha^{(2)}_{ijkl}$ are constant coefficients whose expressions are given by the formula (A.4). For the group SU(2) one has:

$$(\mathrm{d}U)U^{-1}=-2\mathrm{i}f_{ab}(\eta_1,\eta_2,\eta_3)\,\mathrm{d}\eta_b\sigma_a \qquad a,b=1,2,3$$
$$f_{ab}(\eta_1,\eta_2,\eta_3)=\frac{1}{(1+\eta^2)^2}[\delta_{ab}-2\varepsilon_{abc}\eta_c+(\delta_{ad}\delta_{cb}+\delta_{ac}\delta_{db}-\delta_{ab}\delta_{dc})\eta_c\eta_d] \tag{42}$$

which can be derived directly from the parametrization formula (39) or obtained from (41) by imposing $\eta_0=0$.

By the same method we have for the expressions of the Cartan forms in the case of group U(3) parametrized by (38):

$$(\mathrm{d}U)U^{-1}=-2\mathrm{i}F_{ij}(\eta_0,\eta_1,\ldots,\eta_8)\,\mathrm{d}\eta_j\lambda_i$$
$$F_{ij}(\eta_0,\eta_1,\ldots,\eta_8) \tag{43}$$
$$=\frac{\delta_{ij}+\beta^{(1)}_{ijk}\eta_k+\beta^{(2)}_{ijkl}\eta_k\eta_l+\beta^{(3)}_{ijklm}\eta_k\eta_l\eta_m+\beta^{(4)}_{ijklmn}\eta_k\eta_l\eta_m\eta_n}{(1-3\eta_0^2+\eta_s\eta_s)^2+(\sqrt{6}\eta_0-\eta_0^3+\sqrt{6}\eta_0\eta_s\eta_s-\frac{2}{3}\mathrm{i}h_{pqr}\eta_p\eta_q\eta_r)^2}$$

where βs are constant coefficients whose expressions are given by the formulae (A.8). The Cartan forms for the group SU(3) are defined by the same equation (43), but one must take into account the connection equation (40) as an additional equation.

4.3. *The Lagrangians of the principal chiral fields* (PCF) *for the groups* U(2), SU(2), U(3), SU(3)

The Lagrangians of the PCF are defined as the left- and right-invariant metrics in Lie algebras [10, 19]:

$$\mathscr{L}=-\mathrm{const}\,\mathrm{Sp}[(\mathrm{d}U)U^{-1}(\mathrm{d}U)U^{-1}]. \tag{44}$$

The Lagrangian of PCF for the group U(2) has the form:

$$\begin{aligned}\mathscr{L}=\mathrm{const}[(1-\eta_0^2(x)+\eta^2(x))^2+4\eta_0^2(x)]^{-2}[\delta_{jj'}&+(\alpha^{(1)}_{jj'k}+\alpha^{(1)}_{j'jk})\eta_k(x)\\&+(\alpha^{(2)}_{jj'kl}+\alpha^{(2)}_{j'jkl}+\alpha^{(1)}_{ijk}\alpha^{(1)}_{ij'l})\eta_k(x)\eta_l(x)\\&+(\alpha^{(1)}_{ijk}\alpha^{(2)}_{ij'lm}+\alpha^{(1)}_{ij'k}\alpha^{(2)}_{ijlm})\eta_k(x)\eta_l(x)\eta_m(x)\\&+\alpha^{(2)}_{ijkl}\alpha(2)_{ij'mn}\eta_k(x)\eta_l(x)\eta_m(x)\eta_n(x)]\partial_\mu\eta_j(x)\partial^\mu\eta_{j'}(x)\end{aligned} \tag{45}$$

where all the indices take the values 0, 1, 2, 3. The Lagrangian of PCF for the group SU(2) takes the very simple form:

$$\mathscr{L}=\mathrm{const}\frac{(\partial\eta(x))^2}{(1+\eta^2(x))^2}. \tag{46}$$

The Lagrangian of PCF for the group U(3) is

$$\mathscr{L}=\mathrm{const}\,F_{ij}(\eta(x))F_{ij'}(\eta(x))\partial_\mu\eta_j(x)\partial^\mu\eta_{j'}(x) \tag{47}$$

where $F_{ij}(\eta)=F_{ij}(\eta_0,\eta_1,\ldots,\eta_8)$ are defined by (43). For the group SU(3) one must add to (47) the equation (40) regarded as a connection equation.

4.4. *The Lagrangians of Goldstone fields* (GF)

Let us now consider the chiral fields which take values in the coset space G/H. It is well known [7, 11] that in the case of system with spontaneous symmetry breaking, if G is the symmetry group of the Lagrangian, H is the invariant subgroup of the vacuum, then the parameters of the coset space G/H can be identified with GF. Let us denote with V_α the generators of the subgroup H, with A_i the generators which supplement H to the entire group G. The Cartan forms in this case can be written as

$$(\mathrm{d}G(Q))G^{-1}(Q)=\mathrm{i}Q_n'^{(1)}X_n=\mathrm{i}(\theta_\alpha V_\alpha+\omega_i A_i). \tag{48}$$

The differential forms θ_α, ω_i depend, in general case, on v, $\mathrm{d}v$, a, $\mathrm{d}a$, where v and a are, respectively, the parameters of H and G/H. In a geometric interpretation θ_a, ω_i are analogous to the rotations and displacements of a co-frame in the Riemmanian spaces [6, 7]. For setting up the Lagrangians of GF one can consider only the space of parameters a_i of G/H, assuming $v_\alpha=0$, and define the Lagrangian as the square interval of the geodesics between two points a_i and $a_i+\mathrm{d}a_i$ [6, 7]

$$\mathscr{L}=\text{const}\ \omega^i(a,\partial_\mu a)\omega^j(a,\partial^\mu a)C^\alpha_{ik}C^k_{\alpha j}. \tag{49}$$

For defining the covariant derivatives of the fields which interact with GF one uses the forms θ_α [6, 7].

For setting up the Lagrangian of GF which correspond to the breaking of the chiral symmetry $\mathrm{U}(2)\times\mathrm{U}(2)$ to the subgroup $\mathrm{U}(2)$ we proceed from the direct product of two independent groups $\mathrm{U}(2)$ parametrized by the R^4-vectors η_i and χ_j, $i,j=0,1,2,3$ (see formula (37)):

$$G=U(\eta)\otimes U(\chi)\qquad U(\eta)=\frac{1-\mathrm{i}\eta_i\sigma_i}{1+\mathrm{i}\eta_i\sigma_i}\qquad U(\chi)=\frac{1-\mathrm{i}\chi_i\sigma_i}{1+\mathrm{i}\chi_i\sigma_i}. \tag{50}$$

For small η, χ one has

$$G\simeq 1_4-4\mathrm{i}\left[\eta_i\left(\frac{\sigma_i}{2}\otimes 1_2\right)+\chi_j\left(1_2\otimes\frac{\sigma_j}{2}\right)\right]=1_4-4\mathrm{i}[\eta_i X_i^R+\chi_j X_j^L] \tag{51}$$

where

$$X_i^R=\frac{\sigma_i}{2}\otimes 1_2\qquad X_j^L=1_2\otimes\frac{\sigma_j}{2}$$

obey the following commutation relations:

$$[X_i^R,X_k^R]=\mathrm{i}a_{ijk}X_k^R\qquad [X_i^L,X_j^L]=\mathrm{i}a_{ijk}X_k^L\qquad [X_i^L,X_j^R]=0 \tag{52}$$

a_{ijk} are the structure constants of the group U(2) whose values are given by (A.2). From X_i^R, X_j^L we make the combinations $V_\alpha=X_\alpha^R+X_\alpha^L$, $A_i=X_i^R-X_i^L$ which obey now the following commutation relations:

$$\begin{aligned}&[V_\alpha,V_\beta]=\mathrm{i}a_{\alpha\beta\gamma}V_\gamma &&[V_\alpha,A_j]=\mathrm{i}a_{\alpha jk}A_k\\ &[A_i,A_j]=\mathrm{i}a_{ij\alpha}V_\alpha &&(i,j,k=0,1,2,3).\end{aligned} \tag{53}$$

V_α generate the subgroup H, A_i are related to the broken symmetries. From (51) one can write for the expression for an infinitely small element G in V_α, A_i

$$G\simeq 1-2\mathrm{i}[v_\alpha V_\alpha+a_iA_i]\qquad v_\alpha=\eta_\alpha+\chi_\alpha\qquad a_i=\eta_i-\chi_i \tag{54}$$

where v_α, a_i are, respectively, the parameters of subgroup H and the coset space G/H.

The Cartan forms for the group $G=\mathrm{U}(2)\times\mathrm{U}(2)$ can be written as:

$$\begin{aligned}&(\mathrm{d}G)G^{-1}=-4\mathrm{i}(\eta_i'^{(1)}X_i^R+\chi_k'^{(1)}X_k^L)\\ &\eta_i'^{(1)}=f_{ij}(\eta_0,\eta_1,\eta_2,\eta_3)\,\mathrm{d}\eta_j \qquad \chi_k'^{(1)}=f_{kl}(\chi_0,\chi_1,\chi_2,\chi_3)\,\mathrm{d}\chi_l\end{aligned} \tag{55}$$

where $f_{ij}(\eta)$ are given by the formula (41). Transforming (55) to the form:

$$\begin{aligned}&(\mathrm{d}G)G^{-1}=-2\mathrm{i}[\theta_\alpha V_\alpha+\omega_i A_i]\\ &\theta_\alpha=\eta_\alpha'^{(1)}+\chi_\alpha'^{(1)}=f_{\alpha j}\,\mathrm{d}\eta_j+f_{\alpha l}(\chi)\,\mathrm{d}\chi_l\\ &\omega_i=\eta_i'^{(1)}-\chi_i'^{(1)}=f_{ij}(\eta)\,\mathrm{d}\eta_j-f_{il}(\chi)\,\mathrm{d}\chi_l\end{aligned} \tag{56}$$

and restricting it on the spaces of parameters a_i (see formula (54)): $v_\alpha=\eta_\alpha+\chi_\alpha=0$, $\chi_\alpha=-\eta_\alpha$, $a_i=\eta_i-\chi_i=2\eta_i$, one obtains for the expressions of θ_α and ω_i in the parameters a_i of the coset space $\mathrm{U}(2)\times\mathrm{U}(2)/\mathrm{U}(2)$:

$$\theta_\alpha=[f_{\alpha j}(a)-f_{\alpha j}(-a)]\,\mathrm{d}\alpha_j \qquad \omega_i=[f_{ij}(a)+f_{ij}(-a)]\,\mathrm{d}a_j. \tag{57}$$

According to (49) the Lagrangian of GF which correspond to the breaking of the chiral symmetry $\mathrm{U}(2)\times\mathrm{U}(2)$ to the subgroup $\mathrm{U}(2)$ is given by

$$\mathscr{L}=\mathrm{const}\,\omega^i(\xi,\partial_\mu\xi)\omega^j(\xi,\partial^\mu)C^\alpha_{ik}C^k_{\alpha j} \qquad (i,j,k,\alpha=0,1,2,3) \tag{58}$$

where $C^\alpha_{ik}=a_{ik\alpha}$ are the structure constants of the group U(2) (see (A.1))

$$\omega^i(\xi,\partial_\mu\xi)=[f_{ij}(\xi)+f_{ij}(-\xi)]\partial_\mu\xi_j. \tag{59}$$

It is easy to show $C^\alpha_{ak}C^k_{\alpha b}=\delta_{ab}$, therefore one obtains

$$\begin{aligned}\mathscr{L}&=\mathrm{const}\,\omega^a(\xi(x),\partial_\mu\xi(x))\omega^a(\xi(x),\partial^\mu\xi(x))\\ &=\mathrm{const}[(1-\xi_0^2(x)+\xi^2(x))^2+4\xi_0^2(x)]^{-2}\\ &\quad\times\{\delta_{aj}\delta_{aj'}+(\delta_{aj'}\alpha^{(2)}_{ajkl}+\delta_{aj}\delta_{aj'kl})\xi_k(x)\xi_l(x)\\ &\quad+\alpha^{(2)}_{ajkl}\alpha^{(2)}_{aj'mn}\xi_k(x)\xi_l(x)\xi_m(x)\xi_n(x)\}\partial_\mu\xi_j(x)\partial^\mu\xi_{j'}(x)\\ &\quad i,j,k=0,1,2,3 \qquad a=1,2,3.\end{aligned} \tag{60}$$

For the Lagrangian of GF which correspond to the breaking of $\mathrm{SU}(2)\times\mathrm{SU}(2)$ to SU(2) one has

$$\mathscr{L}=\mathrm{const}\frac{[(\partial\xi(x))^2+4(\xi(x)\partial\xi(x))^2-2\xi^2(x)(\partial\xi(x))^2+\xi^4(x)(\partial\xi(x))^2]}{(1+\xi^2(x))^4}. \tag{61}$$

Note that by the change of field variable

$$\pi(x)=2\frac{\xi(x)}{1-\xi^2} \tag{62}$$

the Lagrangian (61) can be brought to the simple form:

$$\mathscr{L}=\mathrm{const}\frac{(\partial\pi(x))^2}{(1+\pi^2(x))^2}. \tag{63}$$

Furthermore, considering the components of the field $\pi(x)$ as stereographic coordinates of the sphere S^3 in the space R^4 and changing them to the coordinates $\varphi^\nu(x)$ of the space $R^4 (\nu=1,2,3,4)$, the Lagrangian (64) takes the familiar form

$$\mathscr{L}=\tfrac{1}{2}\partial_\mu\varphi^\nu(x)\partial^\mu\varphi^\nu(x) \qquad \varphi^\nu(x)\varphi^\nu(x)=1 \tag{64}$$

which is the Lagrangian of the n-field.

In the same way, but by more cumbersome calculations one has the Lagrangian of the GF which which correspond to the breaking U(3) × U(3) to U(3):

$$\begin{aligned}\mathcal{L} = \text{const}\, s^{-2}(\xi(x))\{&\partial_\mu \xi_a(x)\partial^\mu \xi_a(x) + [\gamma^{(2)}_{jj'kl}\xi_k(x)\xi_l(x)\\ &+\gamma^{(4)}_{jj'klmn}\xi_k(x)\xi_l(x)\xi_m(x)\xi_n(x)\\ &+\gamma^{(6)}_{jj'klmnpq}\xi_k(x)\xi_l(x)\xi_m(x)\xi_n(x)\xi_p(x)\xi_q(x)\\ &+\gamma^{(8)}_{jj'klmnpqrs}\xi_k(x)\xi_l(x)\xi_m(x)\xi_n(x)\xi_p(x)\xi_q(x)\xi_r(x)\xi_s(x)]\\ &\times \partial_\mu \xi_j(x)\partial^\mu \xi_{j'}(x)\}\end{aligned} \tag{65}$$

where

$$\begin{aligned}s(x) = (1 - 3\xi_0^2(x) + \xi_i(x)\xi_i(x))^2 - (&\mathrm{i}\sqrt{6}\xi_0(x) - \mathrm{i}\xi_0^3(x)\\ &+\mathrm{i}\sqrt{6}\,\xi_0(x)\xi_i(x)\xi_i(x) + h_{i_1i_2i_3}\xi_{i_1}(x)\xi_{i_2}(x)\xi_{i_3}(x))^2\end{aligned}$$

$\xi_i(x) = (\xi_0(x), \xi_a(x))$, $a = 1, 2, \ldots, 8$; the constant coefficients γ are given by the formula (A.9). The Lagrangian of the GF which correspond to the breaking SU(3) × SU(3) to SU(3) consists of the Lagrangian (65) and the connection equation (40).

5. Conclusion

The method developed in this work for calculating Cartan forms is applicable for any case when one uses a natural (in the sense of the formula (1)) parametrization in which a composition law of finite (but not only infinitesimal) group transformations is given. By using this method and the linear and natural parametrizations of groups the gauge theories of spacetime and inner symmetries can be formulated in a finite (but not only infinitesimal) approach. The gauge field transformations under the finite local group transformations may be useful, for example, in the problem of quantization of gauge theories for eliminating non-physical variables (gauge degrees of freedom) before quantization.

The method gives also a rather effective tool for setting up the Lagrangian of PCFs and GFs. The obtained Lagrangians are distinguished from the previous known ones by the new types of nonlinearity (ratios of polynomials). For the group SU(2), the Lagrangians (see (46) and (62)) are equivalent to the Lagrangian of n-field on the sphere S^3, but may be more convenient due to the fact that all the informations about interaction are contained in the Lagrangian without any additional connection equation. For the groups U(3) and SU(3), the Lagrangians of PCF and GF have not been written elsewhere.

Appendix

The commutators and anticommutators of the matrices $\sigma_i = (\sigma_0, \sigma_a)$, where $\sigma_0 = I$, $\sigma_{1,2,3}$ are the Pauli matrices, have the form

$$[\sigma_i, \sigma_j]_- = 2\mathrm{i}a_{ijk}\sigma_k \qquad \{\sigma_i, \sigma_j\}_+ = 2s_{ijk}\sigma_k \qquad i, j, k = 0, 1, 2, 3 \tag{A.1}$$

where a_{ijk}, s_{ijk}, respectively, are antisymmetric and symmetric in any pair of indices. The non-zero components of a_{ijk}, s_{ijk} are

$$a_{abc} = \varepsilon_{abc} \qquad s_{000} = 1 \qquad s_{0ab} = \delta_{ab} \qquad (a, b, c = 1, 2, 3). \tag{A.2}$$

Denoting

$$h_{ijk} = a_{ijk} - s_{ijk} \qquad (i, j, k = 0, 1, 2, 3) \tag{A.3}$$

one has for the coefficients $\alpha^{(1)}_{ijk}$, $\alpha^{(2)}_{ijkl}$ in the formula (41)

$$\begin{aligned} \alpha^{(1)}_{ijk} &= h_{jki} - h_{kji} \\ \alpha^{(2)}_{ijkl} &= 4\delta_{ij}\delta_{0k}\delta_{0l} + 2i(h_{jki} + h_{kji})\delta_{0l} - h_{kjm}h_{mli}. \end{aligned} \tag{A.4}$$

The commutators and anticommutators of the matrices

$$\lambda_i = (\lambda_0, \lambda_a)$$

($\lambda_0 = \sqrt{\frac{2}{3}}I$, λ_a $(a = 1, 2, \ldots, 8)$ are the Gell-Mann matrices) are

$$[\lambda_i, \lambda_j]_- = 2iA_{ijk}\lambda_k \qquad \{\lambda_i, \lambda_j\}_+ = 2S_{ijk}\lambda_k \qquad (i, j, k = 0, 1, \ldots, 8) \tag{A.5}$$

where A_{ijk}, S_{ijk}, respectively, are antisymmetric and symmetric in any pair of indices. The non-zero components of A_{ijk}, S_{ijk} are:

$$A_{abc} = f_{abc} \qquad S_{000} = \sqrt{\tfrac{2}{3}} \qquad S_{0ab} = \sqrt{\tfrac{2}{3}}\delta_{ab} \qquad S_{abc} = d_{abc} \tag{A.6}$$

where f_{abc}, d_{abc} are defined by the following relations:

$$[\lambda_a, \lambda_b] = 2if_{abc}\lambda_c, \{\lambda_a, \lambda_b\}_+ = 2d_{abc}\lambda_c \qquad (a, b, c = 1, 2, \ldots, 8). \tag{A.7}$$

The values of f_{abc}, d_{abc} are given in any textbook on particle physics. The coefficients h_{ijk}, $\beta^{(1)}$, $\beta^{(2)}$, $\beta^{(3)}$, $\beta^{(4)}$ in formula (643) have the form:

$$\begin{aligned} & h_{ijk} = f_{ijk} - \mathrm{i}d_{ijk} \qquad \beta^{(1)}_{ijk} = h_{jki} - h_{kji} \\ & \beta^{(2)}_{ijkl} = \delta_{ij}\delta_{nk}\delta_{nl} + h_{jkm}h_{mli} - h_{kjm}h_{mli} + h_{klm}h_{mji} \\ & \beta^{(3)}_{ijklm} = 3(h_{jki} - h_{kji})\delta_{0l}\delta_{0m} + (h_{jki} - h_{kji})\delta_{lm} \\ & \qquad + \mathrm{i}\sqrt{6}(h_{jkn}h_{nli} - h_{kln}h_{nji})\delta_{0m} + (h_{kln}h_{njp}h_{pmi} - h_{kjn}h_{nlp}h_{pmi}) \\ & \beta^{(4)}_{ijklmn} = 9\delta_{ij}\delta_{0k}\delta_{0l}\delta_{0m}\delta_{0n} - 6\delta_{ij}\delta_{0k}\delta_{mn} + \delta_{ij}\delta_{kl}\delta_{mn} \\ & \qquad + 3\mathrm{i}\sqrt{6}(h_{jki} + h_{kji})\delta_{0l}\delta_{0m}\delta_{0n} - \mathrm{i}\sqrt{6}(h_{jki} + h_{kji})\delta_{0l}\delta_{mn} \\ & \qquad - 3(2h_{kjp}h_{pli} + h_{klp}h_{pji} + h_{jkp}h_{pli})\delta_{0m}\delta_{0n} + (h_{jkp}h_{pli} + h_{klp}h_{pji})\delta_{mn} \\ & \qquad - \mathrm{i}\sqrt{6}(h_{klp}h_{pjq}h_{qmi} + h_{kjp}h_{plq}h_{qmi})\delta_{0n} + h_{klp}h_{pjq}h_{qmr}h_{rni}. \end{aligned} \tag{A.8}$$

The coefficients $\gamma^{(2)}$, $\gamma^{(4)}$, $\gamma^{(6)}$, $\gamma^{(8)}$ in formula (65) have the form:

$$\begin{aligned} \gamma^{(2)}_{ijkl} &= \sum_{a=1}^{8} 2\delta_{ai}p_{ajkl} \\ \gamma^{(4)}_{ijklmn} &= \sum_{a=1}^{8} (p_{aikl}p_{ajmn} + 2\delta_{ai}q_{ajklmn}) \\ \gamma^{(6)}_{ijklmnpq} &= \sum_{a=1}^{8} 2p_{aikl}q_{ajmnpq} \\ \gamma^{(8)}_{ijklmnpqrs} &= \sum_{a=1}^{8} q_{aiklmn}q_{ajpqrs} \end{aligned} \tag{A.9}$$

where

$$\begin{aligned} & p_{ijkl} = 2\delta_{ij}\delta_{kl} + h_{klr}h_{rji} + h_{jkr}h_{rli} - h_{kjr}h_{rli} \\ & q_{ijklmn} = 9\delta_{ij}\delta_{0k}\delta_{0l}\delta_{mn} - 6\delta_{ij}\delta_{0k}\delta_{0l}\delta_{mn} + \delta_{ij}\delta_{kl}\delta_{mn} \\ & \qquad + 6\sqrt{6}(h_{kji} + h_{jki})\delta_{0l}\delta_{0m}\delta_{0n} - 2\mathrm{i}\sqrt{6}(h_{kji} + h_{jki})\delta_{0l}\delta_{mn} \\ & \qquad - 3(h_{klr}h_{rji} + h_{jkr}h_{rli} - 2h_{kjr}h_{rli})\delta_{0m}\delta_{0n} + (h_{klr}h_{rji} + h_{jkr}h_{rli})\delta_{mn} \\ & \qquad - \mathrm{i}\sqrt{6}(h_{klr}h_{rjs}h_{smi} + h_{kjr}h_{rls}h_{smi})\delta_{0m} + h_{klr}h_{rjs}h_{smt}h_{tni}. \end{aligned}$$

References

[1] Federov F I 1961 *Dokl. Akad. Nauk Byelorus. SSR* **5** 101; 1962 *Dokl Acad. Nauk SSSR* **143** 56
[2] Federov F I 1979 *Lorentz Group* (Moscow: Nauka)
[3] Bogush A A 1980 *Dokl Akad. Nauk Byelorus. SSR* **24** 1073
[4] Berezin A V and Fedorov F I 1982 *Dokl. Akad. Nauk Byeloruss. SSR* **26** 17
[5] Berezin A V, Kuvshinov V I and Fedorov F I 1988 *Dokl. Akad. Nauk SSSR* **302** 317
[6] Cartan E 1949 *Geometry of Lie Group and Symmetric Spaces* (Moscow: Izd. inostr. lit.)
[7] Volkov M K and Pervushin V N 1978 *Essential Nonlinear Quantum Theory, Dynamical Symmetries and Physics of Mesons* (Moscow: Atomizdat)
Volkov M K 1973 *Particles and Nuclei* (Dubna: JINR) **4** 3
[8] Kuvshinov V I and Nguyen Vien Tho 1991 *Dokl. Akad. Nauk Byelorus. SSR* **35** 119
[9] Tseytlin A A 1982 *Phys. Rev.* D **26** 3327
Ponomarev V N, Barvinski A O and Obukhov Yu N 1985 *Geometrodynamical Methods and Gauge Approach to Gravitation* (Moscow: Atomizdat)
[10] Dubrovin B A, Novikov S P and Fomenko A T 1979 *Modern Geometry* (Moscow: Nauka)
[11] Coleman S, Wess J and Zumino B 1969 *Phys. Rev.* **177** 2239
Callan C G, Coleman S, Wess J and Zumino B 1969 *Phys. Rev.* **177** 2247
Salam A and Strathdee J 1969 *Phys. Rev.* **184** 1750
[12] Chang L N and Mansouri F 1978 *Phys. Rev.* D **17** 3168
[13] Gursey F and Marchindon L 1978 *Phys. Rev.* D **17** 2038
[14] Chamseddinne A N and West P C 1977 *Nucl. Phys.* B 129
MacDowel S W and Mansouri F 1977 *Phys. Rev. Lett.* **38** 739
[15] Perelomov A M 1987 *Phys. Rept.* **146** 135; 1989 in *Elementary Particles, Collection of Scientific Works of the Thirteenth School of Physics* ITEP **2** 3 (Moscow: Energoatomizdat)
[16] Skyrme T H R 1961 *Proc. R. Soc.* A **260** 127
[17] Gantmakher P I 1979 *Theory of Matrices* (Moscow: Nauka)
[18] Bogush A A, Fedorovykh A M M and Zhirkov L F 1983 in *Group Theoretical Methods in Physics* **1** 196
[19] Schwartz A S 1989 *Quantum Field Theory and Topology* (Moscow: Nauka)